CALCULATION OF THE NONSTATIONARY TEMPERATURE FIELD IN A COMPOSITE (THREE-LAYER) CYLINDER

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A method of calculating the nonstationary temperature field of a composite hollow cylinder when its external surface is heated asymmetrically is described.

Composite cylinders are widely used in the construction of various thermal devices. In the majority of papers on heat transfer the nonstationary temperature field of a composite hollow cylinder is calculated assuming symmetrical boundary conditions. In practice, the boundary conditions are often not symmetrical. This considerably complicates the boundary-value problem of heat conduction and makes it difficult to obtain a solution using accurate analytical methods.

In the present paper the nonstationary temperature field of a composite (three-layer) hollow cylinder when its external surface is heated asymmetrically is determined using a finite-difference method.

Consider an element in the form of a composite hollow cylinder of infinite length ($-\infty < z < \infty$) of internal radius R_1 and external radius R_4. The materials of the layers D_k ($k = 1, 2, 3$) of the cylinder are different and are characterized by the following thermal and geometrical parameters: λ_k, c_k, ρ_k, and $\delta_s(R_{s-1}, R_s)$ ($s = 2, 3, 4$). The initial temperatures of all the layers are the same, constant, and equal to T_0. At the initial instant of time $t = 0$ the internal surface mnpq of the cylinder and also the parts AB and CD of its external surface are heated at a temperature T_c, which remains constant during the heating. The parts BC and DEA of the external side surface of the cylinder are thermally insulated. There is tight thermal contact between the layers of the cylinders which remains unchanged during the heat transfer. We are required to determine the following:

1) the nonstationary temperature field of the transverse cross section of the composite cylinder;

2) the time taken to heat the zone $R_2 < r < R_3$, $0 \leqslant \varphi \leqslant 2\pi$ of the cylinder to a given temperature T^*;

3) the dependence of the temperature on time at characteristic points of the transverse cross section of the cylinder.

The system of differential equations describing the nonstationary temperature distribution in the three-layer cylinder has the form

$$c_k\rho_k \frac{\partial T_k}{\partial t} = \lambda_k \left(\frac{\partial^2 T_k}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial T_k}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 T_k}{\partial \varphi^2} \right),$$

$$R_{s-1} < r < R_s, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad t > 0, \tag{1}$$

$$T_1(R_1, \varphi, t) = T_c, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad t > 0, \tag{2}$$

$$T_3(R_4, \varphi, t) = T_c, \quad 0 \leqslant \varphi < \frac{\pi}{3}, \quad \frac{5}{6}\pi \leqslant \varphi < \pi, \quad t > 0, \tag{3}$$

$$\frac{\partial T_3(R_4, \varphi, t)}{\partial r} = 0, \quad \frac{\pi}{3} \leqslant \varphi < \frac{5}{6}\pi, \quad \pi \leqslant \varphi \leqslant 2\pi, \quad t > 0, \tag{4}$$

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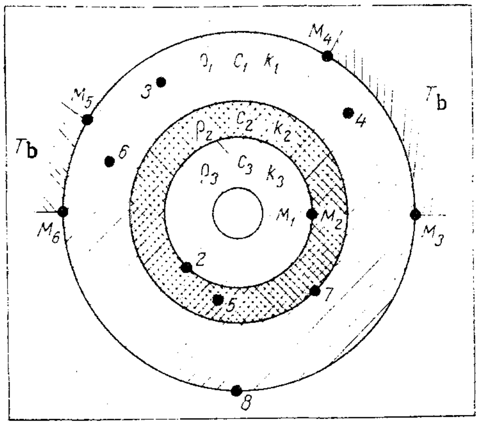

Fig. 1. Transverse cross section of the composite hollow cylinder.

$$T_1(R_2,\ \varphi,\ t) = T_2(R_2,\ \varphi,\ t),\ \lambda_1 \frac{\partial T_1(R_2,\ \varphi,\ t)}{\partial r} = \lambda_2 \frac{\partial T_2(R_2,\ \varphi,\ t)}{\partial r},\ 0 \leqslant \varphi \leqslant 2\pi,\ \ t > 0, \qquad (5)$$

$$T_2(R_3,\ \varphi,\ t) = T_3(R_3,\ \varphi,\ t),\ \ \lambda_2 \frac{\partial T_2(R_3,\ \varphi,\ t)}{\partial r} = \lambda_3 \frac{\partial T_3(R_3,\ \varphi,\ t)}{\partial r},\ 0 \leqslant \varphi \leqslant 2\pi,\ \ t > 0, \qquad (6)$$

$$T(r,\ \varphi,\ 0) = T_0 = \text{const},\ \ R_{s-1} \leqslant r \leqslant R_s,\ \ 0 \leqslant \varphi \leqslant 2\pi,\ \ t = 0. \qquad (7)$$

We will use a finite-difference method to solve the boundary-value problem (1)-(7) numerically. We introduce the space–time net

$$D_{r,\varphi,t} = \{r_i = R_1 + i\Delta r,\ \ \varphi_j = j\Delta\varphi,\ t_n = n\tau\}$$
$$(i = 1,\ 2,\ \ldots,\ M,\ \ j = 1,\ 2,\ \ldots,\ N,\ n = 0,\ 1,\ 2,\ \ldots),$$
$$\Delta r = (R_4 - R_1)/M,\ \ \Delta\varphi = 2\pi/N$$

and we put

$$T(r_i,\ \varphi_j,\ t_n) = T^n_{i,j}.$$

Replacing the derivatives with respect to the coordinates and time from Eqs. (1)-(7) by the corresponding difference relations, we obtain the finite-difference analog of these differential equations in the form [1]

$$T^{n+1}_{i,j} = T^n_{i,j} + \frac{\lambda\tau}{c}\Big[(T^n_{i-1,j} - 2T^n_{i,j} + T^n_{i+1,j})\frac{1}{\Delta r^2} +$$
$$+ (T^n_{i-1,j} - T^n_{i+1,j})\frac{1}{2r_i\Delta r} + (T^n_{i,j+1} - 2T^n_{i,j} + T_{i,j+1})\frac{1}{r_i^2\Delta\varphi}\Big], \qquad (8)$$

$$T_1(R_1,\ \varphi_j,\ t_n) = T_c,\ \ 1 \leqslant j \leqslant N, \qquad (9)$$

$$T_3(R_4,\ \varphi_j,\ t_n) = T_c,\ \ M_3 \leqslant j \leqslant M_4,\ \ M_5 \leqslant j < M_6, \qquad (10)$$

$$T_3(r_{M-1},\ \varphi_j,\ t_n) = T_3(r_M,\ \varphi_j,\ t_n),\ \ M_4 \leqslant j < M_5,\ M_6 \leqslant j < M_3. \qquad (11)$$

Here M_3, M_4, M_5, and M_6 are the values of the index j corresponding to the coordinates of the points of separation of the boundary circles of the cross section of the cylinder (Fig. 1a).

The temperature at the boundary of separation of the cylindrical layer is given by the relations [2] (Fig. 2a)

$$T^{n+1}_{m,m+1} = T^n_{m,m+1} + \frac{2\tau}{\rho_m c_m \Delta r_m + \rho_{m+1}c_{m+1}\Delta r_{m+1}}\left[\frac{\lambda_m}{\Delta r_m}(T^n_m - T^n_{m,m+1}) - \frac{\lambda_{m+1}}{\Delta r_{m+1}}(T^n_{m,m+1} - T^n_{m+1})\right]. \qquad (12)$$

In Eq. (12) the indices i and j are omitted, while the index m, which takes the values 1 and 2, indicates the boundary of separation between layers 1–2 and 2–3, respectively.

Relations (12) follow from the condition that the changes in the amount of heat in the layers adjacent to the boundaries of separation

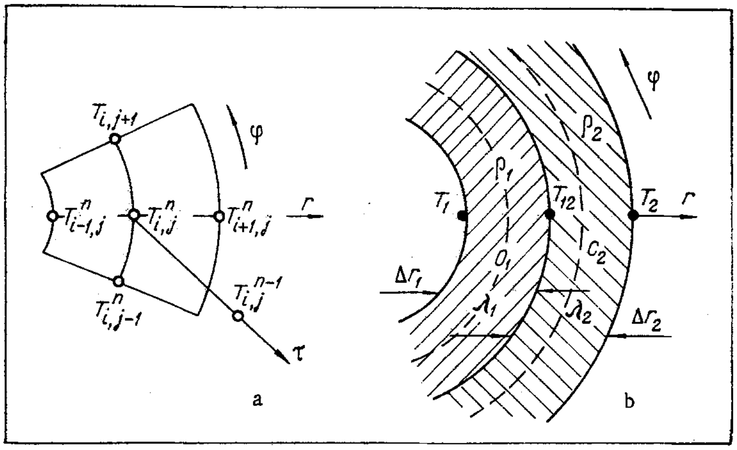

Fig. 2. Scheme for calculating the temperature field using Eq. (8) (a) and for deriving Eq. (12) (b).

$$\Delta Q_{m,m+1} = \left[\frac{\lambda_m}{\Delta r_m} (T^n_m - T^n_{m,m+1}) + \frac{\lambda_{m+1}}{\Delta r_{m+1}} (T^n_{m,m+1} - T^n_{m+1}) \right] \tau \tag{13}$$

are equivalent to changes in their heat content, due to changes in the boundary temperatures from T^n_b to T^{n+1}_b:

$$\Delta Q_{m,m+1} = \left(\rho_m c_m \frac{\Delta r_m}{2} + \rho_{m+1} c_{m+1} \right) (T^{n+1}_{m,m+1} - T^n_{m,m+1}). \tag{14}$$

The initial values of the temperatures at all points of the net are given by the equations

$$T_k(r_i, \varphi_j, 0) = T_0, \; i = 1, 2, \ldots, M, \; j = 1, 2, \ldots, N. \tag{15}$$

An algorithm for the computational process was developed using an explicit difference scheme represented by the recurrent relation (8). The convergence and stability of the computational process based on relation (8) is ensured by a matched choice of the steps of the space–time net: with the chosen steps Δr and $\Delta\varphi$ the integration step with respect to time is chosen from the conditions [2]

$$0 \leqslant \left[1 - 2a\tau \left(\frac{1}{\Delta r^2} + \frac{1}{r_i \Delta\varphi^2} \right) \right] \leqslant 1, \tag{16}$$

$$0 \leqslant \left[1 - \frac{2\tau \left(\dfrac{\lambda_m}{\Delta r_m} + \dfrac{\lambda_{m+1}}{\Delta r_{m+1}} \right)}{\rho_m c_m \Delta r_m + \rho_{m+1} c_{m+1} \Delta r_{m+1}} \right] \leqslant 1. \tag{17}$$

Equation (8) enables one to determine the value of the required temperature function $T(r, \varphi, t)$ at any internal point (r_i, φ_j) on the cross section of the three-layer cylinder at an arbitrary instant of time $t = t_{n+1}$ from the known values of the temperature at points of the theoretical "five-point" (Fig. 2b)

$$(r_{i-1}, \varphi_j), (r_i, \varphi_j), (r_{i+1}, \varphi_j) (r_i, \varphi_{j-1}) \text{ и } (r_i, \varphi_{j+1})$$

at the instant $t = t_n$. The computational process carried out using an ALGOL program consists in successively filling (along the rows) the matrix of values of the temperatures $T_{j,i}$ at each time layer $t = t_n$. At the initial instant of time $t = 0$ all the elements of the matrix $T_{j,i}$ are assumed to be equal to T_0; the column $T_{j,1}$ remains unchanged for any t, while the column $T_{j,M}$ consists of four parts, each of which corresponds to either a heating zone or a zone of thermal insulation on the external surface of the cylinder. When each row of the matrix $T^n_{j,1}$ is filled three zones are distinguished (using logic criteria) corresponding to the three layers of the cross section of the cylinder. The elements of the rows of the matrix $T_{j,i}$ at internal points of each zone are calculated using Eq. (8) taking conditions (9)-(13) and (15) into account. The elements of the rows of the matrix $T_{j,i}$ corresponding to values of the temperature at the boundaries of separation of the layers are calculated using Eqs. (12). The calculations are stopped when the middle layer of the cross section of the cylinder is heated to the assigned temperature.

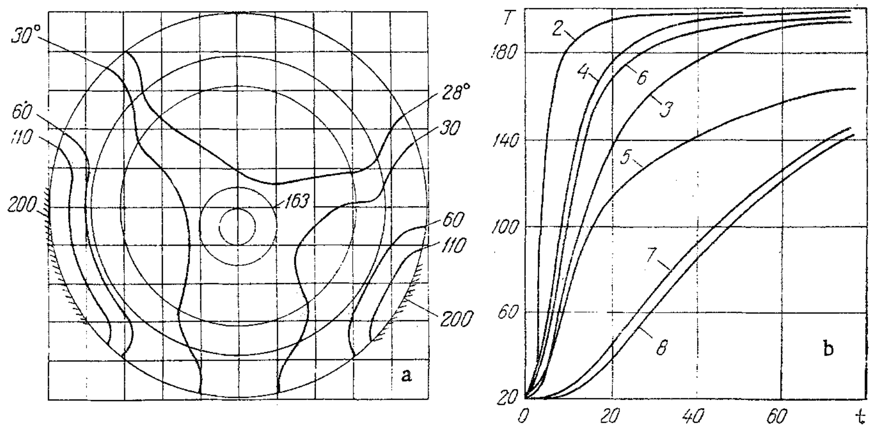

Fig. 3. Pattern of isotherms of the temperature field of the composite cylinder (a) and graphs of T = T(t) at characteristic points of the cross section of the cylinder (b). T, degrees; t, sec.

The nonstationary temperature field of the composite cylinder was calculated on the M-222 computer using the following numerical values of the constructional and thermal parameters:

$$R_1 = 0.02\ \mathrm{m};\quad R_2 = 0.12\ \mathrm{m};\quad R_3 = 0.15\ \mathrm{m};\quad R_4 = 0.2\ \mathrm{m};$$
$$\lambda_1 = \lambda_3 = 36.8\ \mathrm{kcal/m \cdot h \cdot {}^\circ C};\ \lambda_2 = 0.67\ \mathrm{kcal/m \cdot h \cdot {}^\circ C};$$
$$\rho_1 = \rho_3 = 7682\ \mathrm{kg/m^3};\ \rho_2 = 985\ \mathrm{kg/m^3};\ T_0 = 20\ {}^\circ\mathrm{C};$$
$$C_1 = C_3 = 955.2\ \mathrm{kcal/kg \cdot {}^\circ C};\ C_2 = 481\ \mathrm{kcal/kg \cdot {}^\circ C};$$
$$T_c = 300\ {}^\circ\mathrm{C};\ T^* = 140\ {}^\circ\mathrm{C};\ M = 45;\ N = 62;\ \tau = 0.555 \cdot 10^{-4}\ \mathrm{h}.$$

Figure 3a, b shows the pattern of isotherms of the temperature field at the instant t = 0.75r and graphs of T = T(t) at characteristic points of the transverse cross section of the cylinder. The time taken to heat the middle cylindrical layer to a temperature T* = 140°C is 3.8 h.

LITERATURE CITED

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